Removability of time-dependent singularities in the heat equation

Jin Takahashi *† Eiji Yanagida *

Abstract

We consider solutions of the linear heat equation with time-dependent singularities. It is shown that if a singularity is weaker than the order of the fundamental solution of the Laplace equation, then it is removable. We also consider the removability of higher dimensional singular sets. An example of a non-removable singularity is given, which implies the optimality of the condition for removability.

1 Introduction

Removability of singularities of solutions is an interesting and important problem in partial differential equations. For the Laplace equation, the removability of a singular point is defined as follows. Let u be a solution of

$$\Delta u = 0 \quad \text{in } \Omega \setminus \{\xi_0\},$$

where Ω is a domain in $\mathbb{R}^N$ and $\xi_0 \in \Omega$. We say that ξ_0 is a removable singularity if there exists a classical solution $\tilde{u}$ of the Laplace equation such that

$$\tilde{u} \equiv u \quad \text{in } \Omega \setminus \{\xi_0\}.$$

It is well known [3] that for $N \geq 3$, the singular point ξ_0 is removable if and only if

$$|u(x)| = o(|x - \xi_0|^{2-N}) \quad \text{as} \quad x \to \xi_0.$$

For nonlinear elliptic equations, the removability of a singularity has been studied in many papers (see, e.g., Brezis-Veron [1, 4], Veron [11]), and various interesting results have been obtained.

*Department of Mathematics, Tokyo Institute of Technology, Meguro-ku, Tokyo 152-8551, Japan
†E-mail: takahashi.j.ab@m.titech.ac.jp

Similarly, for the heat equation

$$u_t = \Delta u \quad \text{in } \Omega \setminus \{\xi_0\} \times (0,T)$$

with $N \geq 3$ and $T > 0$, Hsu [7] proved recently that the singular point ξ_0 is removable if and only if

$$|u(x,t)| = o(|x-\xi_0|^{2-N}) \quad \text{as} \quad x \to \xi_0$$

for every $t \in (0,T)$. Later, Hui [8] gave a simpler proof for this result. In [6], Hirata extended Hsu and Hui's result to semilinear parabolic equations of the form $u_t = \Delta u + |u|^{p-1}u$ with $p < N/(N-2)$. See also Sato-Yanagida [9] for non-removable singularities in the semilinear parabolic equation.

In this paper, we consider the case where a singular point may move in time and study its removability for the heat equation. More precisely, we formulate our problem as follows. For $T > 0$ fixed, let $\xi : [0,T] \to \mathbb{R}^N$ be a continuous function, and $\Gamma \subset \mathbb{R}^{N+1}$ be a curve given by

$$\Gamma := \{(x,t) \in \mathbb{R}^{N+1} \, : \, x = \xi(t), \; t \in (0,T)\}.$$

We take a domain $\Omega \subset \mathbb{R}^N$ such that $\xi(t) \in \Omega$ for $t \in [0,T]$, and define

$$D := \{(x,t) \in \mathbb{R}^{N+1} \, : \, x \in \Omega \setminus \{\xi(t)\}, \; t \in (0,T)\}.$$

For a solution of

$$u_t = \Delta u \quad \text{in } D, \tag{1.1}$$

the singularity at $x = \xi(t)$ is said to be removable if there exists a function $\tilde{u}$ which satisfies the heat equation in $\Omega \times (0,T)$ in the classical sense and $\tilde{u} \equiv u$ on D.

Our first result gives a condition for the removability of such a (moving) singularity.

Theorem 1.1. *Let $N \geq 3$. Suppose that u satisfies* (1.1) *in the classical sense. Then the singularity of u at $x = \xi(t)$ is removable if and only if for any $0 < t_1 < t_2 < T$ and $0 < \varepsilon < 1$ there exists $0 < r < 1$ depending on t_1, t_2, ε such that*

$$|u(x,t)| \leq \frac{\varepsilon}{|x-\xi(t)|^{N-2}}, \quad 0 < |x-\xi(t)| < r \tag{1.2}$$

for any $t \in [t_1, t_2]$.

Theorem 1.2. *Let $N = 2$. Suppose that u satisfies* (1.1) *in the classical sense. Then the singularity of u at $x = \xi(t)$ is removable if and only if for any $0 < t_1 < t_2 < T$ and $0 < \varepsilon < 1$ the function u satisfies*

$$|u(x,t)| \leq \varepsilon \log \frac{1}{|x-\xi(t)|}, \quad 0 < |x-\xi(t)| < \varepsilon \tag{1.3}$$

for any $t \in [t_1, t_2]$.

Here we note that for $N = 1$, if we define $\tilde{u}$ by

$$\tilde{u}(x,t) := \begin{cases} u(x,t) & \text{for } (x,t) \in D, \\ \liminf\limits_{x\uparrow\xi(t)} u(x,t) & \text{for } (x,t) \in \Gamma, \end{cases}$$

then the singularity at $x = \xi(t)$ is removable if and only if $\tilde{u}$ is continuously differentiable at $x = \xi(t)$ for any $t \in (0,T)$.

Next, we consider a higher dimensional singular set whose spatial codimension is greater than or equal to 2. We reformulate our problem as follows. Let $m \geq 1$, $N \geq m+2$, $T > 0$ and $\mathbf{s} = (s_1, s_2, \ldots, s_m) \in \mathbb{R}^m$. We assume that the mapping

$$\xi(\mathbf{s},t) = (\xi^1(\mathbf{s},t), \xi^2(\mathbf{s},t), \ldots, \xi^N(\mathbf{s},t)) : [0,1]^m \times [0,T] \to \mathbb{R}^N$$

is continuously differentiable with respect to $\mathbf{s}$ and continuous with respect to t. Also, we assume that the Jacobian matrix of ξ with respect to $\mathbf{s}$ is non-singular, that is,

$$\operatorname{rank} \begin{pmatrix} \xi^1_{s_1}(s_1, s_2, \ldots, s_m, t) & \cdots & \xi^1_{s_m}(s_1, s_2, \ldots, s_m, t) \\ \vdots & \ddots & \vdots \\ \xi^N_{s_1}(s_1, s_2, \ldots, s_m, t) & \cdots & \xi^N_{s_m}(s_1, s_2, \ldots, s_m, t) \end{pmatrix} = m \tag{1.4}$$

for any $(s_1, s_2, \ldots, s_m) \in [0,1]^m$ and $t \in [0,T]$. We denote the singular set by

$$\Xi(t) := \{\xi(\mathbf{s},t) \,:\, \mathbf{s} \in [0,1]^m\}$$

and define $\Gamma \subset \mathbb{R}^{N+1}$ by

$$\Gamma := \big\{(x,t) \in \mathbb{R}^{N+1} \,:\, x \in \Xi(t),\ t \in (0,T)\big\}.$$

We also define a distant between x and $\Xi(t)$ by

$$d(x, \Xi(t)) := \min_{\mathbf{s}\in[0,1]^m} |x - \xi(\mathbf{s},t)|.$$

Furthermore, let $\Omega \subset \mathbb{R}^N$ be a domain such that

$$\Omega \supset \bigcup_{t\in[0,T]} \Xi(t),$$

and define a domain $D \subset \mathbb{R}^{N+1}$ by

$$D := \big\{(x,t) \in \mathbb{R}^{N+1} \,:\, x \in \Omega \setminus \Xi(t),\ t \in (0,T)\big\}.$$

Now we define removability of a higher dimensional singular set as follows. For a solution of (1.1), the singular set $\Xi(t)$ is said to be removable if there exists a function $\tilde{u}$ which satisfies the heat equation in $\Omega \times (0,T)$ in the classical sense and $u \equiv \tilde{u}$ on D.

Our results for higher dimensional singular sets are as follows.

Theorem 1.3. *Let $N \geq m+3$. Suppose that ξ satisfies* (1.4) *and that u satisfies* (1.1) *in the classical sense. Then the singular set $\Xi(t)$ is removable if and only if for any $0 < t_1 < t_2 < T$ and $0 < \varepsilon < 1$ there exists $0 < r < 1$ depending on t_1, t_2, ε such that*

$$|u(x,t)| \leq \frac{\varepsilon}{d(x,\Xi(t))^{N-m-2}}, \quad 0 < d(x,\Xi(t)) < r \tag{1.5}$$

for any $t \in [t_1, t_2]$.

Theorem 1.4. *Let $N = m+2$. Suppose that ξ satisfies* (1.4) *and that u satisfies* (1.1) *in the classical sense. Then the singular set $\Xi(t)$ is removable if and only if for any $0 < t_1 < t_2 < T$ and $0 < \varepsilon < 1$ the function u satisfies*

$$|u(x,t)| \leq \varepsilon \log \frac{1}{d(x,\Xi(t))}, \quad 0 < d(x,\Xi(t)) < \varepsilon$$

for any $t \in [t_1, t_2]$.

By an analogous method to Section 3, we can extend Theorem 1.3 to the case where the singular set consists of $\Xi_1, \Xi_2, \ldots, \Xi_k$, each of which satisfies (1.4) and may intersect with others. By regarding $\Xi_1, \Xi_2, \ldots, \Xi_k$ as local coordinates, the above theorems give a condition for the removability in the case where the singular set is a compact m-dimensional C^1-manifold in $\mathbb{R}^N$.

Next, we show the existence of a solution of (1.1) whose singularity moves in time and is not removable. Again, let $N \geq 2$, $T > 0$, and $\Gamma \subset \mathbb{R}^{N+1}$ be defined as above. Moreover, by some technical reason, we assume that $\xi : [0,T] \to \mathbb{R}^N$ is Lipschitz continuous.

The next result implies that the conditions in Theorem 1.1 and Theorem 1.2 for the removability are optimal in some sense.

Theorem 1.5. *Given any Lipschitz continuous function $\xi(t) : [0,T] \to \mathbb{R}^N$, there exists u defined on a neighborhood of Γ such that u satisfies* (1.1) *in the classical sense but the singularity of u at $x = \xi(t)$ is not removable.*

In Section 4, we give an example of a non-removable moving singularity. In fact, this theorem will be proved by solving the following problem:

$$u_t - \Delta u = \delta(x - \xi(t)) \quad \text{in } \mathbb{R}^N \times (0,T), \tag{1.6}$$

where $\delta(\cdot)$ denote the Dirac distribution concentrated at the point $0 \in \mathbb{R}^N$. In this case, we can show that the singularity at $x = \xi(t)$ persists for $(0,T)$ and the solution satisfies

$$\begin{aligned} u(x,t) &= C_1(t)|x-\xi(t)|^{2-N} + o(|x-\xi(t)|^{2-N}) &\quad \text{if} \quad N \geq 3, \\ u(x,t) &= C_2(t)\log(|x-\xi(t)|) + o\left(\log|x-\xi(t)|\right) &\quad \text{if} \quad N = 2 \end{aligned}$$

at $x = \xi(t)$ with some positive bounded continuous functions $C_1(t)$ and $C_2(t)$.

This paper is organized as follows. In Section 2 we prove Theorems 1.1 and 1.2 by cutting a neighborhood of the singularity. In Section 3 we apply this method to a higher dimensional singular set. Section 4 is devoted to the analysis of (1.6).

2 Removability of a moving singularity

In this section, we consider removability of a moving singularity.

Proof of Theorem 1.1. Necessity is easily proved by the same argument as in Section 3 of [7]. Indeed, if the singularity of u at $x = \xi(t)$ is removable, then u is bounded near $x = \xi(t)$.

We prove sufficiency. Let $0 < t_1 < t_2 < T$ and $0 < \varepsilon < 1$. By our assumption, there exists $r = r(t_1, t_2, \varepsilon) > 0$ such that (1.2) holds. For each $t \in (0, T)$, we take any sequence $\{x_i(t)\}_{i=1}^{\infty} \subset \Omega \setminus \{\xi(t)\}$ such that $|x_i(t) - \xi(t)| \to 0$ as $i \to \infty$, and set

$$\tilde{u}(x,t) := \begin{cases} u(x,t) & \text{for } (x,t) \in D, \\ \liminf_{i\to\infty} u(x_i(t), t) & \text{for } (x,t) \in \Gamma. \end{cases}$$

Our goal is to prove that $\tilde{u}$ satisfies heat equation in $\Omega \times (0, T)$ in the classical sense.

First, we show $\tilde{u} \in L^1_{\mathrm{loc}}(\Omega \times (0,T))$. For each $t \in [t_1, t_2]$, we denote

$$B(\xi(t), r) := \{x \in \mathbb{R}^N \,:\, |x - \xi(t)| < r\}.$$

By N-dimensional polar coordinates centered at $\xi(t)$, we have

$$\int_{t_1}^{t_2} \int_{B(\xi(t),r)} |x - \xi(t)|^{2-N} \, dxdt = C_1(t_2 - t_1) r^2 \tag{2.1}$$

for some $C_1 = C_1(N) > 0$. Let $K \subset \mathbb{R}^N$ be a compact subset of Ω. Since $\xi(t) \in \Omega$ for $t \in [0, T]$, we can take $r = r(t_1, t_2, \varepsilon) > 0$ so small that $B(\xi(t), r) \subset \Omega$ for every $t \in [t_1, t_2]$. By (1.2) and (2.1), there exists $C_2 > 0$ such that

$$\begin{aligned} \int_{t_1}^{t_2} \int_K |\tilde{u}(x,t)| \, dxdt \le & \int_{t_1}^{t_2} \int_{K \setminus B(\xi(t,r))} |u(x,t)| \, dxdt + \varepsilon \int_{t_1}^{t_2} \int_{B(\xi(t),r)} |x - \xi(t)|^{2-N} \, dxdt \\ \le & C_2 + \varepsilon C_1 (t_2 - t_1) r^2 < \infty. \end{aligned}$$

Since $0 < t_1 < t_2 < T$ are arbitrary, we have $\tilde{u} \in L^1_{\mathrm{loc}}(\Omega \times (0,T))$.

Next, we show that $\tilde{u}$ satisfies heat equation in $\Omega \times (0, T)$ in the distribution sense. For this purpose, let $\phi \in C_0^\infty(\Omega \times (0,T))$ be a test function, and take a family of cut-off functions $\{\eta_r\}_{r>0} \subset C^\infty(\Omega \times (0,T))$ such that

$$\eta_r(x,t) = \begin{cases} 0 & \text{if} \quad |x - \xi(t)| < r/2, \\ 1 & \text{if} \quad |x - \xi(t)| > r, \end{cases}$$

and

$$0 \le \eta_r \le 1, \quad |\nabla \eta_r| \le C_3/r, \quad |\Delta \eta_r| \le C_3/r^2, \quad |(\eta_r)_t| \le C_3/r, \tag{2.2}$$

where $C_3 > 0$ is a constant independent of t. Since $\eta_r\phi \in C_0^\infty(\Omega \times (0,T))$, and $\tilde{u}$ is a classical solution of (1.1), we have

$$\begin{aligned}\int_\Omega \tilde{u}(x,t_2)\phi(x,t_2)\eta_r(x,t_2) - \tilde{u}(x,t_1)\phi(x,t_1)\eta_r(x,t_1)\,dx \\ = \int_{t_1}^{t_2}\int_\Omega \tilde{u}\{(\phi\eta_r)_t + \Delta(\phi\eta_r)\}\,dxdt.\end{aligned} \tag{2.3}$$

Here, we claim that the following convergence properties hold:

$$\limsup_{\varepsilon\to 0}\left|\int_{t_1}^{t_2}\int_\Omega \tilde{u}\Delta\phi\,dxdt - \int_{t_1}^{t_2}\int_\Omega \tilde{u}\Delta(\phi\eta_r)\,dxdt\right| = 0, \tag{2.4}$$

$$\limsup_{\varepsilon\to 0}\left|\int_{t_1}^{t_2}\int_\Omega \tilde{u}\phi_t\,dxdt - \int_{t_1}^{t_2}\int_\Omega \tilde{u}(\phi\eta_r)_t\,dxdt\right| = 0, \tag{2.5}$$

$$\limsup_{\varepsilon\to 0}\left|\int_\Omega \tilde{u}(x,t_1)\phi(x,t_1)\,dx - \int_\Omega \tilde{u}(x,t_1)\phi(x,t_1)\eta_r(x,t_1)\,dx\right| = 0, \tag{2.6}$$

$$\limsup_{\varepsilon\to 0}\left|\int_\Omega \tilde{u}(x,t_2)\phi(x,t_2)\,dx - \int_\Omega \tilde{u}(x,t_2)\phi(x,t_2)\eta_r(x,t_2)\,dx\right| = 0. \tag{2.7}$$

To show (2.4), we rewrite

$$\begin{aligned}&\int_{t_1}^{t_2}\int_\Omega \tilde{u}\Delta\phi\,dxdt - \int_{t_1}^{t_2}\int_\Omega \tilde{u}\Delta(\phi\eta_r)\,dxdt \\ &\quad= \int_{t_1}^{t_2}\int_\Omega \tilde{u}(1-\eta_r)\Delta\phi\,dxdt - 2\int_{t_1}^{t_2}\int_\Omega \tilde{u}\nabla\phi\cdot\nabla\eta_r\,dxdt - \int_{t_1}^{t_2}\int_\Omega \tilde{u}\phi\Delta\eta_r\,dxdt \\ &\quad=: I_{1,r} - 2I_{2,r} - I_{3,r}.\end{aligned} \tag{2.8}$$

By (1.2) and (2.2), for sufficiently small $r = r(t_1,t_2,\varepsilon) > 0$, we have the inequalities

$$\begin{aligned}|I_{1,r}| &\le \|\Delta\phi\|_{L^\infty(\Omega\times(0,T))}\varepsilon\int_{t_1}^{t_2}\int_{B(\xi(t),r)}|x-\xi(t)|^{2-N}\,dxdt, \\ |I_{2,r}| &\le \|\nabla\phi\|_{L^\infty(\Omega\times(0,T))}C_3\frac{\varepsilon}{r}\int_{t_1}^{t_2}\int_{B(\xi(t),r)}|x-\xi(t)|^{2-N}\,dxdt, \\ |I_{3,r}| &\le \|\phi\|_{L^\infty(\Omega\times(0,T))}C_3\frac{\varepsilon}{r^2}\int_{t_1}^{t_2}\int_{B(\xi(t),r)}|x-\xi(t)|^{2-N}\,dxdt.\end{aligned}$$

Hence, by (2.1) and $r \in (0,1)$, we have

$$\begin{aligned}|I_{1,r}| &\le \|\Delta\phi\|_{L^\infty(\Omega\times(0,T))}C_1(t_2-t_1)\varepsilon r^2 \le C_4\varepsilon, \\ |I_{2,r}| &\le \|\nabla\phi\|_{L^\infty(\Omega\times(0,T))}C_1C_3(t_2-t_1)\varepsilon r \le C_4\varepsilon, \\ |I_{3,r}| &\le \|\phi\|_{L^\infty(\Omega\times(0,T))}C_1C_3(t_2-t_1)\varepsilon \le C_4\varepsilon\end{aligned}$$

for some $C_4 > 0$. Hence we obtain (2.4). Similarly we obtain (2.5), (2.6) and (2.7) from above estimates.

Thus, the function $\tilde{u}$ satisfies

$$\int_\Omega \tilde{u}(x,t_2)\phi(x,t_2) - \tilde{u}(x,t_1)\phi(x,t_1)\,dx = \int_{t_1}^{t_2}\int_\Omega \tilde{u}(\phi_t + \Delta\phi)\,dxdt \tag{2.9}$$

for any $\phi \in C_0^\infty(\Omega \times (0,T))$. Since $0 < t_1 < t_2 < T$ be arbitrary, the function $\tilde{u} \in L^1_{\mathrm{loc}}(\Omega \times (0,T))$ satisfies the heat equation in $\Omega \times (0,T)$ in the distribution sense. By using the Weyl lemma for the heat equation (see, e.g., Section 6 of [5] or [10]), $\tilde{u}$ satisfies the heat equation in $\Omega \times (0,T)$ in the classical sense. Since $\tilde{u} = u$ in D, the singularity of u at $x = \xi(t)$ is removable. ∎

Proof of Theorem 1.2. We prove only sufficiency. Let $0 < t_1 < t_2 < T$ and $0 < \varepsilon < 1$, and define $\tilde{u}$ as in the proof of Theorem 1.1. By 2-dimensional polar coordinates, we have

$$\int_{t_1}^{t_2}\int_{B(\xi(t),\varepsilon)} \log\frac{1}{|x-\xi(t)|}\,dxdt \le C_1(t_2-t_1)\left(1+\log(1/\varepsilon)\right)\varepsilon^2 \tag{2.10}$$

for some $C_1 > 0$. This implies $\tilde{u} \in L^1_{\mathrm{loc}}(\Omega \times (0,T))$.

We show that $\tilde{u}$ satisfies heat equation in $\Omega \times (0,T)$ in the distribution sense. Let $\phi \in C_0^\infty(\Omega \times (0,T))$, and take $\{\eta_\varepsilon\}_{\varepsilon>0}^\infty \subset C^\infty(\Omega \times (0,T))$ such that

$$\eta_\varepsilon(x,t) = \begin{cases} 0 & \text{if} \quad |x-\xi(t)| < \varepsilon/2, \\ 1 & \text{if} \quad |x-\xi(t)| > \varepsilon, \end{cases}$$

and

$$0 \le \eta_\varepsilon \le 1, \quad |\nabla\eta_\varepsilon| \le C_2/\varepsilon, \quad |\Delta\eta_\varepsilon| \le C_2/\varepsilon^2, \quad |(\eta_\varepsilon)_t| \le C_2/\varepsilon \tag{2.11}$$

for some $C_2 > 0$. Since $\tilde{u}$ satisfies (1.1), the equality (2.3) holds for $r = \varepsilon$. Again, we claim that the convergence properties (2.4), (2.5), (2.6) and (2.7) hold for $r = \varepsilon$. Let $I_{1,\varepsilon}, I_{2,\varepsilon}$ and $I_{3,\varepsilon}$ be defined as in (2.8) with $r = \varepsilon$. By (1.3) and (2.11), for sufficiently small $\varepsilon > 0$, we have

$$\begin{aligned}
|I_{1,\varepsilon}| &\le \|\Delta\phi\|_{L^\infty(\Omega\times(0,T))}\varepsilon\int_{t_1}^{t_2}\int_{B(\xi(t),\varepsilon)} \log\frac{1}{|x-\xi(t)|}\,dxdt, \\
|I_{2,\varepsilon}| &\le \|\nabla\phi\|_{L^\infty(\Omega\times(0,T))}C_2\int_{t_1}^{t_2}\int_{B(\xi(t),\varepsilon)} \log\frac{1}{|x-\xi(t)|}\,dxdt, \\
|I_{3,\varepsilon}| &\le \|\phi\|_{L^\infty(\Omega\times(0,T))}C_2\frac{1}{\varepsilon}\int_{t_1}^{t_2}\int_{B(\xi(t),\varepsilon)} \log\frac{1}{|x-\xi(t)|}\,dxdt.
\end{aligned}$$

Hence by (2.10), we have

$$\begin{aligned}
|I_{1,\varepsilon}| &\le \|\Delta\phi\|_{L^\infty(\Omega\times(0,T))}C_1(t_2-t_1)\left(1+\log(1/\varepsilon)\right)\varepsilon^3 \le C_3\varepsilon\log(1/\varepsilon), \\
|I_{2,\varepsilon}| &\le \|\nabla\phi\|_{L^\infty(\Omega\times(0,T))}C_1C_2(t_2-t_1)\left(1+\log(1/\varepsilon)\right)\varepsilon^2 \le C_3\varepsilon\log(1/\varepsilon), \\
|I_{3,\varepsilon}| &\le \|\phi\|_{L^\infty(\Omega\times(0,T))}C_1C_2(t_2-t_1)\left(1+\log(1/\varepsilon)\right)\varepsilon \le C_3\varepsilon\log(1/\varepsilon)
\end{aligned}$$

for some $C_3 > 0$. Hence we obtain (2.4). Similarly we obtain (2.5), (2.6) and (2.7) from above estimates. These imply that $\tilde{u} \in L^1_{\rm loc}(\Omega \times (0,T))$ satisfies the heat equation in $\Omega \times (0,T)$ in the distribution sense. The remainder is the same as in the proof of Theorem 1.1. ∎

3 Removability of a singular set

Let $\Xi(t) \subset \mathbb{R}^N$, $D \subset \mathbb{R}^{N+1}$, $\Gamma \subset \mathbb{R}^{N+1}$, and $\Omega \subset \mathbb{R}^N$ are the sets defined in Section 1. To show Theorems 1.3 and 1.4, we give the following estimates.

Lemma 3.1. *There exists $C_1 = C_1(N,m) > 0$ and $C_2 = C_2(m) > 0$ such that for every sufficiently small $r > 0$,*

$$\int_{A_{r,t}} d(x,\Xi(t))^{m+2-N}\,dx \le C_1 r^2 \qquad \text{if} \quad N \ge m+3, \tag{3.1}$$

$$\int_{A_{r,t}} \log \frac{1}{d(x,\Xi(t))}\,dx \le C_2 r^2 \left(1 + \log\frac{1}{r}\right) \qquad \text{if} \quad N = m+2 \tag{3.2}$$

for any $t \in (0,T)$, where $A_{r,t} := \{x \in \mathbb{R}^N : d(x,\Xi(t)) < r\}$.

Proof. We prove the lemma only in the case $N \ge m+3$. In fact, (3.2) can be proved in the same manner as (3.1). Let $t \in (0,T)$ be fixed. We extend the domain of the function ξ to $[a,b]^m \times [0,T]$ with $a < 0$ and $b > 1$. That is, we take a mapping

$$\tilde{\xi}(\mathbf{s},t) = (\tilde{\xi}^1(\mathbf{s},t), \tilde{\xi}^2(\mathbf{s},t), \dots, \tilde{\xi}^N(\mathbf{s},t)) : [a,b]^m \times [0,T] \to \mathbb{R}^N$$

such that $\tilde{\xi}$ is continuously differentiable in $\mathbf{s}$ and continuous in t. In addition, we assume that $\tilde{\xi}$ satisfies (1.4) and

$$\tilde{\xi}\big|_{[0,1]^m\times[0,T]} = \xi, \quad \tilde{\xi}^j_{s_i}\big|_{[0,1]^m\times[0,T]} = \xi^j_{s_i}, \quad i = 1,2,\dots,m, \quad j = 1,2,\dots,N.$$

We define

$$\widetilde{\Xi}(t) := \{\tilde{\xi}(\mathbf{s},t) \,:\, \mathbf{s} \in [a,b]^m\}.$$

For each $\mathbf{s} \in (a,b)^m$, let $\Pi_{r,t}(\mathbf{s})$ be a subset of a normal plane of $\widetilde{\Xi}(t)$ at $\tilde{\xi}(\mathbf{s},t)$ given by

$$\Pi_{r,t}(\mathbf{s}) := \{x \in A_{r,t} \,:\, (x - \tilde{\xi}(\mathbf{s},t)) \cdot \tilde{\xi}_{s_i}(\mathbf{s},t) = 0 \ \text{ for any } i = 1,2,\dots,m\}.$$

Since $\tilde{\xi}(\cdot,t)$ is defined on a compact set, there exists a sufficiently small $r > 0$ such that

$$d(x,\widetilde{\Xi}(t)) = |x - \tilde{\xi}(t)|, \quad x \in \Pi_{r,t}(\mathbf{s}) \tag{3.3}$$

for each $\mathbf{s} \in (a,b)^m$. Again by compactness, we have

$$M := \max_{t\in[0,T]} \int_{\tilde{\Xi}(t)} d\sigma^m < \infty, \tag{3.4}$$

where $d\sigma^m$ is an m-dimensional surface element. Since $\tilde{\xi}$ satisfies (1.4), $\Pi_{r,t}(\mathbf{s})$ is a part of an $(N-m)$-dimensional hyperplane. For each $\mathbf{s} \in (a,b)^m$, let $P_{\mathbf{s}} : \mathbb{R}^N \to \mathbb{R}^N$ be a congruent transformation defined by

$$P_{\mathbf{s}} x = (y_1, y_2, \ldots, y_{N-m}, 0, \ldots, 0), \quad x \in \Pi_{r,t}(\mathbf{s})$$

for some $y_1, y_2, \ldots, y_{N-m} \in \mathbb{R}$. Now, by using $(N-m)$-dimensional polar coordinates, we obtain

$$\int_{P_{\mathbf{s}}(\Pi_{r,t}(\mathbf{s}))} |y|^{m+2-N}\, dy_1 dy_2 \cdots dy_{N-m} = C_3 \int_0^r \rho^{m+2-N+(N-m-1)}\, d\rho = C_4 r^2, \tag{3.5}$$

where C_3, $C_4 > 0$ depend on N, m but not on $\mathbf{s}, t$.

Recall that the congruent transformations preserve a distance between any two points and that the function $\tilde{\xi}$ is an extension of ξ. Hence by choosing sufficiently small $r > 0$ again if necessary, we have the estimate

$$\int_{A_{r,t}} d(x, \xi(\cdot,t))^{m+2-N}\, dx \le M C_4 r^2$$

by using (3.3), (3.4) and (3.5). Thus we obtain (3.1). ∎

Proof of Theorem 1.3. We adopt the same approach as in the proof of Theorem 1.1, so we state the outline only.

Let $0 < t_1 < t_2 < T$ and $0 < \varepsilon < 1$. By our assumption, there exists $r = r(t_1, t_2, \varepsilon) > 0$ such that (1.5) holds. For $t \in (0,T)$, we take any sequence $\{x_i(t)\}_{i=1}^\infty \subset \Omega \setminus \Xi(t)$ such that $d(x_i(t), \xi(\cdot,t)) \to 0$ as $i \to \infty$, and set

$$\tilde{u}(x,t) := \begin{cases} u(x,t) & \text{for } (x,t) \in D, \\ \liminf\limits_{i\to\infty} u(x_i(t),t) & \text{for } (x,t) \in \Gamma. \end{cases}$$

By Lemma 3.1, we obtain $\tilde{u} \in L^1_{\mathrm{loc}}(\Omega\times(0,T))$. We show that $\tilde{u}$ satisfies (1.1) in $\Omega\times(0,T)$ in the distribution sense. Let $\phi \in C_0^\infty(\Omega \times (0,T))$, and take $\{\eta_r\}_{r>0}^\infty \subset C^\infty(\Omega \times (0,T))$ such that

$$\eta_r(x,t) = \begin{cases} 0 & \text{if} \quad d(x,\xi(\cdot,t)) < r/2, \\ 1 & \text{if} \quad d(x,\xi(\cdot,t)) > r, \end{cases}$$

and η_r satisfies the condition (2.2) for some $C > 0$. Since $\tilde{u}$ satisfies (1.1), we have (2.3). By Lemma 3.1 and an argument similar to Section 2, we obtain (2.9). That is, the function $\tilde{u} \in L^1_{\mathrm{loc}}(\Omega \times (0,T))$ satisfies the heat equation in $\Omega \times (0,T)$ in the distribution sense. The remainder is the same as in the proof of Theorem 1.1. ∎

Since (3.2) holds, we can show Theorem 1.4 in the same way. We omit details of the proof.

4 Non-removable singularity

In this section, we consider the case where a singularity move in time and is not removable. Without loss of generality, we take $\Omega = \mathbb{R}^N$. Let $N \geq 2$ and $T > 0$. We assume that $\xi : [0,T] \to \mathbb{R}^N$ is arbitrarily given Lipschitz continuous function with a Lipschitz constant $L > 0$.

To show Theorem 1.5, we solve the equation (1.6). In this paper, we say that u satisfies (1.6) in the distribution sense if u belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^N \times (0,T))$ and satisfies

$$\int_0^T \int_{\mathbb{R}^N} (-\phi_t - \Delta\phi) u \, dxdt = \int_0^T \phi(\xi(t), t) \, dt \tag{4.1}$$

for any $\phi \in C_0^\infty(\mathbb{R}^N \times (0,T))$. Now, we denote by

$$\Phi(x,t) := (4\pi t)^{-N/2} \exp(-|x|^2/4t)$$

the fundamental solution of the heat equation. Moreover, we define F in $\mathbb{R}^N \times (0,T)$ by

$$F(x,t) := \int_0^t \Phi(x - \xi(s), t-s) \, ds.$$

In the following, we will show that F satisfies (1.6) in the distribution sense. In addition, we will give upper and lower estimates of F, and we will see that F is an example of Theorem 1.5.

Proposition 4.1. *The function F satisfies* (1.1) *in the classical sense.*

To show Proposition 4.1, we give the following lemma.

Lemma 4.1. *The function F satisfies* (1.6) *in the distribution sense.*

Proof. First, we show $F \in L^1_{\mathrm{loc}}(\mathbb{R}^N \times (0,T))$. By simple calculation, we have

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^N} F(x,t) \, dxdt &= \int_0^T \int_0^t \left(\int_{\mathbb{R}^N} \Phi(x - \xi(s), t-s) \, dx \right) dsdt \\ &= \int_0^T \int_0^t dsdt = \frac{1}{2} T^2 < \infty,\end{aligned}$$

so that $F \in L^1(\mathbb{R}^N \times (0,T))$. In particular, F belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^N \times (0,T))$.

Next, we show that F satisfies (4.1). For this purpose, let $\phi \in C_0^\infty(\mathbb{R}^N \times (0,T))$ be a test function. For each $t \in (0,\tau)$, we take $\tau \in (0,t)$ and define F^τ by

$$F^\tau(x,t) = \int_0^{t-\tau} \Phi(x - \xi(s), t-s) \, ds.$$

Here F^τ is bounded for each fixed τ, that is, there exist $C_1(N)$, $C_2(N) > 0$ such that

$$0 \le F^\tau(x,t) \le C_1(N) \int_0^{t-\tau} (t-s)^{-N/2}\, ds \le C_2(N) \tau^{\frac{2-N}{2}}$$

for each $t \in (0,T)$. Then, integrating by parts yields

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^N} (-\phi_t - \Delta\phi) F^\tau \, dxdt \\
&\quad = \int_0^T \int_{\mathbb{R}^N} (-\phi_t - \Delta\phi) \left(\int_0^{t-\tau} \Phi(x-\xi(s), t-s)\, ds \right) dxdt \\
&\quad = \int_0^T \int_{\mathbb{R}^N} \phi(x,t) \Phi(x - \xi(t-\tau), \tau)\, dxdt \\
&\qquad + \int_0^T \int_{\mathbb{R}^N} \phi(x,t) \left(\int_0^{t-\tau} \{ \Phi_t(x-\xi(s), t-s) - \Delta\Phi(x-\xi(s), t-s) \}\, ds \right) dxdt \\
&\quad = \int_0^T \int_{\mathbb{R}^N} \phi(x,t) \Phi(x-\xi(t-\tau), \tau)\, dxdt.
\end{aligned}$$

Similarly from Section 2.3.1 of [2], we see that

$$\lim_{\tau \to 0} \int_{\mathbb{R}^N} \phi(x,t) \Phi(x - \xi(t-\tau), \tau)\, dx = \phi(\xi(t), t) \tag{4.2}$$

for each $t \in (0,T)$.

For the reader's convenience, we give a proof of (4.2). Let $0 < t < T$ and $\varepsilon > 0$ be fixed. We choose $\delta > 0$ such that

$$|\phi(x,t) - \phi(\xi(t), t)| < \varepsilon \tag{4.3}$$

for any $|x - \xi(t)| < \delta$. Then, we have

$$\begin{aligned}
&\left| \int_{\mathbb{R}^N} \phi(x,t)\Phi(x-\xi(t-\tau),\tau)\, dx - \phi(\xi(t),t) \right| \\
&\quad \le \int_{\mathbb{R}^N} |\phi(x,t) - \phi(\xi(t),t)| \Phi(x - \xi(t-\tau), \tau)\, dx \\
&\quad = \int_{B(\xi(t),\delta)} + \int_{\mathbb{R}^N \setminus B(\xi(t),\delta)} =: I_1 + I_2.
\end{aligned}$$

First, by (4.3), we have an estimate of I_1 as

$$I_1 \le \varepsilon \int_{\mathbb{R}^N} \Phi(x - \xi(t-\tau), \tau)\, dx = \varepsilon.$$

Next, we give an estimate of I_2. If $|x-\xi(t)| \geq \delta$ and $|\xi(t)-\xi(t-\tau)| \leq \delta/2$, then

$$|x-\xi(t)| \leq |x-\xi(t-\tau)| + |\xi(t-\tau)-\xi(t)| \leq |x-\xi(t-\tau)| + \frac{1}{2}|x-\xi(t)|$$

Hence $|x-\xi(t-\tau)| \geq |x-\xi(t)|/2$. By simple calculation,

$$\begin{aligned}
I_2 &\leq 2\|\phi\|_{L^\infty(\mathbb{R}^N\times(0,T))} \int_{\mathbb{R}^N\setminus B(\xi(t),\delta)} (4\pi\tau)^{-N/2} \exp\left(-\frac{|x-\xi(t-\tau)|^2}{4\tau}\right) dx \\
&\leq C_3\tau^{-N/2} \int_{\mathbb{R}^N\setminus B(\xi(t),\delta)} \exp\left(-\frac{|x-\xi(t)|^2}{16\tau}\right) dx \\
&= C_4\tau^{-N/2} \int_\delta^\infty r^{N-1} \exp\left(-\frac{r^2}{16\tau}\right) dr \\
&= C_5 \int_{\delta/4\sqrt{\tau}}^\infty \sigma^{N-1} e^{-\sigma^2}\, d\sigma \to 0 \quad \text{as} \quad \tau \to 0,
\end{aligned}$$

where C_3, C_4, $C_5 > 0$ are constants independent of τ, and $r = 4\sqrt{\tau}\sigma$. Therefore, if we have $|\xi(t)-\xi(t-\tau)| \leq \delta/2$ and take $\tau > 0$ is sufficiently small, then we obtain $I_1 + I_2 \leq \varepsilon$. Thus it is shown that (4.2) holds.

From (4.2) and the Lebesgue theorem, we see that F satisfies (4.1), that is,

$$\int_0^T \int_{\mathbb{R}^N} (-\phi_t - \Delta\phi) F(x,t)\, dxdt = \int_0^T \phi(\xi(t),t)\, dt. \tag{4.4}$$

Hence the function F satisfies (1.6) in the distribution sense. ∎

Proof of Proposition 4.1. Let $\psi \in C_0^\infty(D)$ be a test function, in particular, $\psi \in C_0^\infty(\mathbb{R}^N \times (0,T))$. By (4.4), we have

$$\int_0^T \int_{\mathbb{R}^N} (-\psi_t - \Delta\psi) F\, dxdt = \int_0^T \psi(\xi(t),t)\, dt.$$

Since $\psi(\xi(t),t) = 0$ for any $t \in (0,T)$, we obtain

$$\int_0^T \int_{\mathbb{R}^N} (-\psi_t - \Delta\psi) F\, dxdt = 0.$$

Hence $F \in L^1(\mathbb{R}^N \times (0,T))$ satisfies the heat equation in D in the distribution sense. By the Weyl lemma for the heat equation, we conclude that F satisfies (1.1) in the classical sense. ∎

Proposition 4.2. *Let $N \geq 3$. There exists $C_1 = C_1(L,N,T) > 0$ and $C_2 = C_2(L,N,T) > 0$ such that $F(x,t)$ satisfies*

$$C_1 \frac{1}{|x-\xi(t)|^{N-2}} \leq F(x,t) \leq C_2 \frac{1}{|x-\xi(t)|^{N-2}}$$

for each $t \in (0,T)$ and $0 < |x-\xi(t)| < \sqrt{t}$.

Proof. Let $t \in (0,T)$ be fixed. We set $z := x - \xi(t)$ and assume $0 < |z| < \sqrt{t}$. By changing variable $t - s = |z|^2/(4\sigma)$, we have

$$\begin{aligned} F(x,t) &= C_3(N) \int_0^t (t-s)^{-N/2} \exp\left(-\frac{|\xi(t) + z - \xi(s)|^2}{4(t-s)} \right) ds \\ &= C_4(N)\, |z|^{2-N} \int_{|z|^2/4t}^{\infty} \sigma^{\frac{N}{2}-2} \exp\left(-\sigma \frac{|z + \xi(t) - \xi(s)|^2}{|z|^2} \right) d\sigma \end{aligned}$$

with some constants $C_3(N)$, $C_4(N) > 0$. By simple calculation, we have

$$|z + \xi(t) - \xi(s)|^2 = |z|^2 + 2z \cdot (\xi(t) - \xi(s)) + |\xi(t) - \xi(s)|^2.$$

In order to give an estimate of F, we consider the integral

$$I(z,t) := \int_{|z|^2/4t}^{\infty} \sigma^{\frac{N}{2}-2} e^{-\sigma} \exp\left(-2\sigma \frac{z \cdot (\xi(t) - \xi(s))}{|z|^2} - \sigma \frac{|\xi(t) - \xi(s)|^2}{|z|^2} \right) d\sigma.$$

By the Cauchy-Schwarz inequality and Lipschitz continuity of ξ, we have

$$\begin{aligned} \exp\left(-2\sigma \frac{z \cdot (\xi(t) - \xi(s))}{|z|^2} \right) &\le \exp\left(\frac{2\sigma}{|z|^2} |z| \left| \xi(t) - \xi\Big(t - \frac{|z|^2}{4\sigma}\Big) \right| \right) \\ &\le e^{L|z|/2} \le e^{L\sqrt{t}/2} \le e^{L\sqrt{T}/2} \end{aligned}$$

and

$$\exp\left(-\sigma \frac{|\xi(t) - \xi(s)|^2}{|z|^2} \right) \le 1.$$

Similarly,

$$\exp\left(-2\sigma \frac{z \cdot (\xi(t) - \xi(s))}{|z|^2} \right) \ge e^{-L|z|/2} \ge e^{-L\sqrt{T}/2}.$$

Again by using Lipschitz continuity of ξ, we have

$$\exp\left(-\sigma \frac{|\xi(t) - \xi(s)|^2}{|z|^2} \right) \ge \exp\left(-\frac{\sigma}{|z|^2} \Big(L \frac{|z|^2}{4\sigma} \Big)^2 \right) \ge \exp\left(-L^2 \frac{|z|^2}{16\sigma} \right).$$

Moreover, by substituting $\sigma = |z|^2/(4t)$ into $\exp(-L^2|z|^2/(16\sigma))$,

$$\exp\left(-L^2 \frac{|z|^2}{16\sigma} \right) \ge e^{-L^2 t/4} \ge e^{-L^2 T/4}.$$

Thus we obtain

$$e^{-L\sqrt{T}/2} e^{-L^2T/4} \le \exp\left(-2\sigma \frac{z \cdot (\xi(t) - \xi(s))}{|z|^2} - \sigma \frac{|\xi(t) - \xi(s)|^2}{|z|^2} \right) \le e^{L\sqrt{T}/2}$$

for each $0 < |z| < \sqrt{t}$.

Using these inequalities, $I(z,t)$ is estimated as

$$e^{-L\sqrt{T}/2}e^{-L^2T/4}\int_{|z|^2/4t}^{\infty}\sigma^{\frac{N}{2}-2}e^{-\sigma}\,d\sigma \le I(z,t) \le e^{L\sqrt{T}/2}\int_{|z|^2/4t}^{\infty}\sigma^{\frac{N}{2}-2}e^{-\sigma}\,d\sigma.$$

Hence there exist $C_5(N,L,T),\ C_6(N,L,T) > 0$ such that

$$C_5|z|^{2-N}\int_{|z|^2/4t}^{\infty}\sigma^{\frac{N}{2}-2}e^{-\sigma}\,d\sigma \le F(x,t) \le C_6|z|^{2-N}\int_{|z|^2/4t}^{\infty}\sigma^{\frac{N}{2}-2}e^{-\sigma}\,d\sigma \tag{4.5}$$

for each $0 < |z| < \sqrt{t}$. Here, by direct calculation, we have

$$J(z,t) := \int_{|z|^2/4t}^{\infty}\sigma^{\frac{N}{2}-2}e^{-\sigma}\,d\sigma \le \int_0^{\infty}\sigma^{\frac{N}{2}-2}e^{-\sigma}\,d\sigma = C_7(N)$$

for some $C_7(N) > 0$. On the other hand, since $|z| < \sqrt{t}$, we have $(1/4,\infty) \subset (|z|^2/(4t),\infty)$. Thus, there exists $C_8(N) > 0$ such that

$$J(z,t) \ge \int_{1/4}^{\infty}\sigma^{\frac{N}{2}-2}e^{-\sigma}\,d\sigma = C_8(N).$$

Consequently, we obtain

$$C_8(N) \le J(z,t) \le C_7(N).$$

By these inequalities and (4.5), the proof is complete. ∎

Proposition 4.3. *Let $N = 2$. There exist $C_1 = C_1(L,T) > 0$ and $C_2 = C_2(L,T) > 0$ such that $F(x,t)$ satisfies*

$$C_1\log\frac{1}{|x-\xi(t)|} \le F(x,t) \le C_2\log\frac{1}{|x-\xi(t)|} \tag{4.6}$$

for each $t \in (0,T)$ and $0 < |x-\xi(t)| < \min\{1/(4t), 4t, 1/4\}$.

Proof. We fix $t \in (0,T)$ and set $z := x - \xi(t)$. Setting $N = 2$ in (4.5), we have

$$C_3(L,T)\int_{|z|^2/4t}^{\infty}\sigma^{-1}e^{-\sigma}\,d\sigma \le F(x,t) \le C_4(L,T)\int_{|z|^2/4t}^{\infty}\sigma^{-1}e^{-\sigma}\,d\sigma$$

for each $0 < |z| < \min\{1/(4t), 4t, 1/4\}$ and some $C_3(L,T)$, $C_4(L,T) > 0$. We note

$$|z|^3 < |z|^2/(4t) < |z| < 1/4. \tag{4.7}$$

Since $(|z|^3, \infty) \supset (|z|^2/(4t), \infty)$ by (4.7), we have

$$\begin{aligned}\int_{|z|^2/4t}^{\infty} \sigma^{-1} e^{-\sigma}\, d\sigma &\le \int_{|z|^3}^{\infty} \sigma^{-1} e^{-\sigma}\, d\sigma \\ &\le \int_{|z|^3}^{1} \sigma^{-1}\, d\sigma + \int_{1}^{\infty} e^{-\sigma}\, d\sigma \\ &= 3\log(1/|z|) + e^{-1}.\end{aligned}$$

Hence by $e^{-1} < \log 4 < \log(1/|z|)$, F satisfies

$$F(x,t) \le 4C_4(L,T)\log\left(1/|z|\right),$$

so that the second inequality in (4.6) holds. On the other hand, since $(|z|, 1) \subset (|z|^2/(4t), \infty)$ by (4.7), we have

$$\int_{|z|^2/4t}^{\infty} \sigma^{-1} e^{-\sigma}\, d\sigma \ge \int_{|z|}^{1} \sigma^{-1} e^{-\sigma}\, d\sigma \ge e^{-1}\log(1/|z|).$$

Hence F satisfies

$$F(x,t) \ge e^{-1} C_3(L,T)\log(1/|z|),$$

so that the first inequality in (4.6) holds. ∎

Now Theorem 1.5 immediately follows from Propositions 4.2, 4.3 and Proposition 4.1.

Acknowledgements. The second author was partially supported by JSPS KAKHENHI Grant-in-Aid for Challenging Exploratory Research (No. 22604020).

References

[1] H. Brézis and L. Véron, *Removable singularities for some nonlinear elliptic equations*, Arch. Rational Mech. Anal. **75** (1980/81), no. 1, 1–6.

[2] L. C. Evans, *Partial differential equations*, second ed., Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI, 2010.

[3] G. B. Folland, *Introduction to partial differential equations*, second ed., Princeton University Press, Princeton, NJ, 1995.

[4] B. Gidas and J. Spruck, *Global and local behavior of positive solutions of nonlinear elliptic equations*, Comm. Pure Appl. Math. **34** (1981), no. 4, 525–598.

[5] A. Grigor'yan, *Heat kernel and analysis on manifolds*, AMS/IP Studies in Advanced Mathematics, vol. 47, American Mathematical Society, Providence, RI, 2009.

[6] K. Hirata, *Removable singularities of semilinear parabolic equations*, preprint.

[7] S. Y. Hsu, *Removable singularities of semilinear parabolic equations*, Adv. Differential Equations **15** (2010), no. 1-2, 137–158.

[8] K. M. Hui, *Another proof for the removable singularities of the heat equation*, Proc. Amer. Math. Soc. **138** (2010), no. 7, 2397–2402.

[9] S. Sato and E. Yanagida, *Solutions with moving singularities for a semilinear parabolic equation*, J. Differential Equations **246** (2009), no. 2, 724–748.

[10] D. W. Stroock, *Weyl's lemma, one of many*, Groups and analysis, London Math. Soc. Lecture Note Ser., vol. 354, Cambridge Univ. Press, Cambridge, 2008, pp. 164–173.

[11] Laurent Véron, *Singularities of solutions of second order quasilinear equations*, Pitman Research Notes in Mathematics Series, vol. 353, Longman, Harlow, 1996.